University of Bahrain

SERIAL

College of Information Technology
Department of Computer Science

ITCS253 Discrete Structures II

Second Semester 2015/2016

First Exam – One Hour

************ Key Solution ************

STUDENT NAME	**** **Key Solution** ****
STUDENT#	**** **Key Solution** ****
SECTION	**** **Key Solution** ****

- This exam contains **5 pages** (including this cover page) and **4 questions**. Check to see if any pages are missing.
- You are **allowed** to use Calculators.
- You are **not allowed** to use books, notes, or mobiles.
- Please write **one answer**. In case of writing multiple answers, mistakes in any answer will be counted.

Question	Points	Score
1	8	
2	8	
3	8	
4	6	
Total:	30	

Instructor: Dr. Ali Alsaffar Sections# 1 & 2

Answer all questions

(1) Answer the following questions.

(a) [**2 points**] Let $\mathbf{A} = \{x \in \mathbf{R} \mid x \geq a\}$ and define $f : \mathbf{A} \to \mathbf{R}$ by $f(x) = \sqrt{\log x}$. What is the smallest value of a such that f is a function, where a is a real number. Explain your answer.

> **Solution:** We know that for $\log x$, $x > 0$ and for $\sqrt{x}$, $x \geq 0$. However, for $0 < x < 1$, $(\log x) < 0$. Hence, $x \geq 1$ and the smallest value of $a = 1$.

(b) [**2 points**] Define $f : \mathbf{Z} \to \mathbf{N}$ by

$$f(x) = \begin{cases} 1/x, & \text{if } x \geq 0 \\ \sqrt{|x|}, & \text{if } x < 0 \end{cases}$$

Suppose $f(x) = 0$, what is the value of x (if it exists.) Explain your answer.

> **Solution:** Either $f(x) = 1/x = 0$ or $f(x) = \sqrt{|x|} = 0$. In both cases x does not exist because for $1/x = 0$, there is no solution to x and for $f(x) = \sqrt{|x|} = 0$, x is 0 but 0 is not negative because $x < 0$.

(c) [**2 points**] Let $6a_{n-7} + 2a_{n-5} - 4a_{n-6} = 0$ be a recurrence relation with $n \geq 7$. Write a_n in terms of its predecessors $a_{n-1}, a_{n-2}, a_{n-3}, \ldots$ and what is its order?

> **Solution:** $6a_{n-7} + 2a_{n-5} - 4a_{n-6} = 0$
> $\implies 6a_{n-2} + 2a_n - 4a_{n-1} = 0$
> $\implies 3a_{n-2} + a_n - 2a_{n-1} = 0$
> $\implies a_n = 2a_{n-1} - 3a_{n-2}$
> and the order is two.

(d) [**2 points**] Let $S = \mathbf{R}^+$ and $a * b = c$, where $c < a + b$. Is * a binary operation on S? Justify your answer.

> **Solution:** No. Because for $a = 8, b = 10$, then $c < 8 + 10 \implies c < 18$ and c is not unique.

(2) (a) **[4 points]** Let $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(n) = (n-1)(n+3)$. Determine if f is one-to-one or not.

Solution: For any $n, m \in \mathbf{Z}$, assume $f(n) = f(m)$.
$\therefore (n-1)(n+3) = (m-1)(m+3)$
$\implies n^2 + 2n - 3 = n^2 + 2n - 3$
$\implies n^2 - m^2 = -2(n-m)$
$\implies (n-m)(n+m) = -2(n-m)$.
Case #1: $n = m$, we are done.
Case #2: $n \neq m \implies n + m = -2$, and this equation has many solutions. For example, $n = -5$ and $m = 3$, then $f(-5) = f(3)$ but $-5 \neq 3$.

(b) **[4 points]** Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = \dfrac{3x^2}{2x^2+1}$. Find the range.

Solution:
Let $y = \dfrac{3x^2}{2x^2+1}$. Solve for x,
$\implies y(2x^2+1) = 3x^2$
$\implies 3x^2 - 2yx^2 = y$
$\implies x^2 = \dfrac{y}{3-2y}$
$\implies x = \pm\sqrt{\dfrac{y}{3-2y}} \quad (y \neq 3/2)$.
Now, y exists only if $\dfrac{y}{3-2y} \geq 0 \implies y(3-2y) \geq 0$.
Case #1: $y \geq 0 \wedge 3 - 2y \geq 0 \implies y \geq 0 \wedge y \leq 3/2 \implies 0 \leq y < 3/2$ (because $\neq 3/2$).
Case #2: $y \leq 0 \wedge 3 - 2y \leq 0 \implies y \leq 0 \wedge y \geq 3/2$ which is not possible
$\therefore$ The range is $\{y \in \mathbf{R} \mid 0 \leq y < 3/2\}$.

(3) (a) [**3 points**] Suppose G is a group and e is the identity element. If $a^{-1} * b^{-1} * a * b = e$, show that G is an abelian for any $a, b \in G$.

Solution: $a^{-1} * b^{-1} * a * b = e$ left multiply both sides by a
$\Longrightarrow a * a^{-1} * b^{-1} * a * b = a * e$
$\Longrightarrow b^{-1} * a * b = a$ left multiply both sides by b
$\Longrightarrow b * b^{-1} * a * b = b * a$
$\Longrightarrow a * b = b * a$

(b) Let S be a set of three elements given by $S = \{a, b, c\}$ and assume the operation * defined on S is described by the table below.

$*$	a	b	c
a	c	a	b
b	a	b	c
c	c	a	c

(1) [**2 points**] Is the binary operation* commutative? Why?

Solution: No. Because the entries are not symmetric with respect to the main diagonal.
Also, $b * c = c$ and $c * b = a$. Hence, $b * c \neq c * b$ and * is not commutative.

(2) [**1 point**] Determine whether there is an identity element in S for *.

Solution: There is no identity element.

(3) [**2 points**] If there is an identity element, find the inverse for each element.

Solution: Since there is no identity element, then the inverse for each element is indeterministic (can not be determined).

(4) [6 points] Use the Iteration Method (*Back Substitution*) to solve the following recurrence relation.

$$a_0 = 1, \quad a_n = 3a_{n-1} - 1, \quad n \geq 1$$

Solution:

$$
\begin{aligned}
a_n &= 3a_{n-1} - 1 \\
&= 3[3a_{n-2} - 1] - 1 && \text{substitute} \\
&= 3^2 a_{n-2} - 3 - 1 && \text{simplify} \\
&= 3^2[3a_{n-3} - 1] - 3 - 1 && \text{substitute} \\
&= 3^3 a_{n-3} - 3^2 - 3 - 1 && \text{simplify} \\
&= 3^3[a_{n-4} - 1] - 3^2 - 3 - 1 && \text{substitute} \\
&= 3^4 a_{n-4} - 3^3 - 3^2 - 3 - 1 && \text{simplify} \\
&\dots \\
&\text{at step } k \\
&= 3^k a_{n-k} - \sum_{j=0}^{k-1} 3^j \\
&= 3^k a_{n-k} - \frac{3^k - 1}{3 - 1} \qquad (1) \\
&= \dots \\
&\text{until } a_{n-k} = a_0 \implies n - k = 0 \implies n = k, \text{ substitute in (1)} \\
a_n &= 3^n a_0 - \frac{1}{2} \cdot 3^n + \frac{1}{2}
\end{aligned}
$$

$\therefore a_n = \frac{1}{2}(3^n + 1).$